

On the role of dealing with quantum coherence in amplitude amplification

Alexey E. Rastegin

Department of Theoretical Physics, Irkutsk State University, Gagarin Bv. 20, Irkutsk 664003, Russia

Amplitude amplification is one of primary tools in building algorithms for quantum computers. This technique develops key ideas of the Grover search algorithm. The original formulation by Grover has been reformulated in order to make building blocks of the algorithm as generally as possible. Potentially useful modifications are connected with changing phases in the rotation operations and replacing the intermediate Hadamard transform with arbitrary unitary one. In addition, arbitrary initial distribution of the amplitudes may be prepared. There are practical problems, in which we may have *a priori* information about the database. We examine trade-off relations between measures of quantum coherence and the success probability in amplitude amplification processes. We try to understand how prior knowledge and other modifications of algorithm blocks should be exploited properly. As measures of coherence, the geometric coherence and the relative entropy of coherence are mainly considered. In terms of the relative entropy of coherence, complementarity relations with the success probability seem to be the most expository. The general relations presented are illustrated within several model scenarios of amplitude amplification process.

Keywords: Grover's algorithm, quantum search, relative entropy, geometric coherence

I. INTRODUCTION

Grover's search algorithm [1–3] is one of fundamental discoveries that are underlying quantum computations [4]. It was soon recognized that Grover's algorithm is optimal for searching by queries to oracle [5, 6]. That is, we invoke the oracle to evaluate any item, whereas database *per se* is not represented explicitly. Today, there exist a class of quantum algorithms inspired by Grover's search algorithm [7]. Due to the broad applicability of Grover's algorithm [8], researchers have attempted to formulate building blocks of the algorithm as generally as possible. The original algorithm may be modified by changing phases in the rotation operations and replacing the intermediate Hadamard transform with arbitrary unitary. In addition, quantum computing may begin with an arbitrary initial distribution of the amplitudes. The results of analyzing generalized versions of Grover's algorithm were described in [9–11]. Similar technique is one of the basic tool in amplitude amplification.

It is well known that entanglement is a key resource in quantum information processing. The quantum parallelism of Deutsch [12] assumes the use to entangled states of a quantum register. The authors of [13] argued that if entanglement is absent in a quantum algorithm, then the algorithm can be classically simulated with an equivalent amount of classical resources. On the other hand, quantum computations are closely connected with only one or few prescribed bases, in which states of the register are naturally represented. In order to analyze the quantum computational speed-up, we should think about quantum correlations in the context of the concrete computational basis. The problem of quantifying coherence at the quantum level is closely related is currently the subject of active researches [14, 15]. Studies of the role of quantum coherence in performing quantum computations were reported in [16, 17]. It seems that the coherence should be viewed as a key resource for increasing the power of quantum algorithms. The authors of [17] reported on coherence depletion in the original Grover algorithm.

The aim of this work is to study trade-off relations between quantum coherence and the success probability in amplitude amplification. We develop and extend the scope of [17] in the following respects. First, more general computing scenarios are addressed. Second, general relations between quantum coherence and the success probability will be formulated. Third, the geometric coherence will be utilized in the algorithmic context. The paper is organized as follows. In Sect. II, we review the required material on coherence quantifiers and techniques of amplitude amplification like Grover's algorithm. General relations between quantum coherence and the success probability are considered in Sect. III. In particular, we derive a two-sided estimate on the relative entropy of coherence in terms of the success probability. In Sect. IV, the presented trade-off relations are exemplified within some model scenarios of amplitude amplification. In Sect. V, we conclude the paper with a summary of the results obtained. Necessary results of solving recursion equations for a generalized version of Grover's algorithm are gathered in Appendix A.

II. PRELIMINARIES

In this section, we recall definitions of quantum coherence quantifiers that will be used through the paper. We further describe a generalized version of the Grover search algorithm. Studying the role of quantum coherence in the context of amplitude amplification, we will typically referred to the calculation basis. Let the quantum register

contain n qubits. Then the basis is formed by $N = 2^n$ orthonormal kets. Each ket $|x\rangle$ of the calculational basis is indexed by binary n -string with $x_j \in \{0, 1\}$. A rigorous framework for the quantification of coherence was proposed in [14]. We consider the set \mathcal{I} of all diagonal density matrices written as

$$\boldsymbol{\delta} = \sum_{x=0}^{N-1} \delta_x |x\rangle\langle x|. \quad (1)$$

One further asks how far the given state is from those states that are completely incoherent in this basis. The authors of [14] listed general conditions for quantifiers of coherence. Additional restrictions imposed on coherence measures were discussed in [15]. In the present paper, we will mainly use the relative entropy of coherence and the geometric coherence.

Using the quantum relative entropy as a measure of distinguishability, we arrive at the relative entropy of coherence. For density matrices $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$, the quantum relative entropy is expressed as [18]

$$D_1(\boldsymbol{\rho}||\boldsymbol{\sigma}) := \begin{cases} \text{tr}(\boldsymbol{\rho} \ln \boldsymbol{\rho} - \boldsymbol{\rho} \ln \boldsymbol{\sigma}), & \text{if } \text{ran}(\boldsymbol{\rho}) \subseteq \text{ran}(\boldsymbol{\sigma}), \\ +\infty, & \text{otherwise.} \end{cases} \quad (2)$$

By $\text{ran}(\boldsymbol{\rho})$, we mean here the range of $\boldsymbol{\rho}$. The authors of [14] defined the coherence measure

$$C_1(\boldsymbol{\rho}) := \min_{\boldsymbol{\delta} \in \mathcal{I}} D_1(\boldsymbol{\rho}||\boldsymbol{\delta}). \quad (3)$$

Taking the minimization, we finally obtain [14]

$$C_1(\boldsymbol{\rho}) = S_1(\boldsymbol{\rho}_{\text{diag}}) - S_1(\boldsymbol{\rho}). \quad (4)$$

Here, the matrix $\boldsymbol{\rho}_{\text{diag}} = \text{diag}(\langle x|\boldsymbol{\rho}|x\rangle)$ is the diagonal part and $S_1(\boldsymbol{\rho}) = -\text{tr}(\boldsymbol{\rho} \ln \boldsymbol{\rho})$ is the von Neumann entropy of $\boldsymbol{\rho}$. The first part of the right-hand side of (4) reads the Shannon entropy calculated with probabilities $p(x) = \langle x|\boldsymbol{\rho}|x\rangle$, namely

$$S_1(\boldsymbol{\rho}_{\text{diag}}) = H_1(p) := -\sum_{x=0}^{N-1} p(x) \ln p(x). \quad (5)$$

The maximal value of the latter is equal to $\ln N$. It then follows from (4) that the relative entropy of coherence does not exceed $\ln N$. Basic properties of the quantity (3) are discussed in [14, 15]. Uncertainty relations in terms of the relative entropy of coherence were derived in [19–21]. Other coherence monotones could be considered. Coherence monotones of the Tsallis type were examined in [22]. Relative Rényi entropies of coherence were studied in [23, 24]. However, these quantifiers do not reduce to a simple form similar to the expression (4).

Several distance-based quantifiers of coherence were considered [14, 15]. The ℓ_1 -norm of coherence is often used as intuitively natural quantifier. For the given state $\boldsymbol{\rho}$, we define

$$C_{\ell_1}(\boldsymbol{\rho}) := \min_{\boldsymbol{\delta} \in \mathcal{I}} \|\boldsymbol{\rho} - \boldsymbol{\delta}\|_{\ell_1} = \sum_{x \neq y} |\langle x|\boldsymbol{\rho}|y\rangle|. \quad (6)$$

Using the quantity (6), complementarity and uncertainty relations for quantum coherence were studied in [25, 26]. In the context of amplitude amplification, however, the ℓ_1 -norm of coherence seems to not very convenient. Even in simple cases relations between this coherence quantifier and the success probability are sufficiently complicated. Another intuitive way is to use the squared ℓ_2 -norm. However, the corresponding coherence quantifier does not satisfy the monotonicity requirement [14]. The trace norm also induces an interesting candidate for quantification of coherence [27, 28].

We will use the geometric coherence which is introduced in terms of the quantum fidelity [29, 30]. The fidelity of density matrices $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$ is expressed as

$$F(\boldsymbol{\rho}, \boldsymbol{\sigma}) = \|\sqrt{\boldsymbol{\rho}} \sqrt{\boldsymbol{\sigma}}\|_1^2, \quad (7)$$

where $\|\mathbf{A}\|_1 = \text{tr}(\sqrt{\mathbf{A}^\dagger \mathbf{A}})$ is the trace norm. This definition is due to Jozsa [30]. The fidelity is sometimes defined as the square root of (7) [18]. The fidelity ranges between 0 and 1 taking the value 1 for two identical states. Using the 1 minus fidelity as a distance measure, one defines the geometric coherence by [15]

$$C_g(\boldsymbol{\rho}) := 1 - \max_{\boldsymbol{\delta} \in \mathcal{I}} F(\boldsymbol{\rho}, \boldsymbol{\delta}). \quad (8)$$

Properties of this coherence quantifier are summarized in subsection III.C.3 of [15]. In the case of pure state, the formula (8) reduces to

$$C_g(|\psi\rangle) = 1 - \max_x |\langle x|\psi\rangle|^2. \quad (9)$$

It will be shown that the geometric coherence is useful in studies of tradeoff between coherence and the success probability in amplitude amplification. Using the 1 minus the square root of (7), we also have a distance measure. The corresponding coherence quantifier was studied in [14, 27]. Other coherence quantifiers were considered in the literature [14, 15].

Let us recall some results concerning generalizations of Grover's algorithm. In our calculations, we will follow the scheme of analysis developed in [9, 10]. Suppose that the search space contains $N = 2^n$ items indexed by binary n -string $x = (x_1 \cdots x_n)$ with $x_j \in \{0, 1\}$ so that $x \in \{0, 1, \dots, N-1\}$. The problem is to find one of marked items which form the set \mathcal{M} . In amplitude amplification, we try to increase maximally amplitudes of states $|x\rangle$ just for $x \in \mathcal{M}$. Without loss of generality, the number of marked items is assumed to obey $1 \leq |\mathcal{M}| \leq N/2$.

Checking each concrete item is realized through an oracle-computable Boolean function $x \mapsto F(x)$ such that $F(x) = 1$ for $x \in \mathcal{M}$ and $F(x) = 0$ for $x \in \mathcal{M}^c$. The original Grover algorithm starts with initializing n -qubit register to $|0\rangle$ and applying the Hadamard transform to get a uniform amplitude distribution

$$\mathbf{H}|0\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle. \quad (10)$$

Following [9, 10], we will use phase rotations of two different classes. Rotations of the first class non-trivially act on unknown marked states. For an arbitrary phase γ , we define

$$\mathbf{J}_F(\gamma) = \sum_{x=0}^{N-1} \exp(i\gamma F(x)) |x\rangle\langle x|. \quad (11)$$

Thus, the operation (11) rotates all the marked states by the phase γ . For $\gamma = \pi$, we have $|x\rangle \mapsto -|x\rangle$ for $F(x) = 1$. This phase rotation is used in the original Grover algorithm. Its realization with invoking the oracle black box is well explained in the literature. Rotations of the second class rotate some prescribed state $|s\rangle$ of the bases. For any phase β , one reads

$$\mathbf{J}_s(\beta) = \mathbb{1} - (1 - \exp(i\beta)) |s\rangle\langle s|. \quad (12)$$

where $\mathbb{1}$ is the identity operator. So, the ket $|s\rangle$ is rotated by β , whereas other basis kets remain unchanged. One of the well known ingredients of Grover's algorithm can be written as

$$- \mathbf{J}_0(\pi) = 2|0\rangle\langle 0| - \mathbb{1}. \quad (13)$$

Clothing (13) by the Hadamard transforms, we obtain the inversion about the average.

In the original formulation, the Grover iteration is written as

$$\mathbf{G}_0 = -\mathbf{H} \mathbf{J}_0(\pi) \mathbf{H} \mathbf{J}_F(\pi). \quad (14)$$

For several reasons, generalizations of Grover's original algorithm have been developed. In the generalized formulation, the operation (14) is replaced by

$$\mathbf{G} = -\mathbf{U} \mathbf{J}_s(\beta) \mathbf{U}^\dagger \mathbf{J}_F(\gamma), \quad (15)$$

with arbitrary β and γ . Here, the Hadamard transform on n qubits is replaced by an arbitrary unitary operator. To the given $|s\rangle$ and \mathbf{U} , we assign $|\eta\rangle = \mathbf{U}|s\rangle$, whence

$$- \mathbf{U} \mathbf{J}_s(\beta) \mathbf{U}^\dagger = (1 - \exp(i\beta)) |\eta\rangle\langle \eta| - \mathbb{1}. \quad (16)$$

So, the effects of both \mathbf{U} and $|s\rangle$ is summarized in a single term $|\eta\rangle$. After t iterations, the state of the quantum register is represented as

$$|g(t)\rangle = \sum_{x \in \mathcal{M}} k_x(t) |x\rangle + \sum_{y \in \mathcal{M}^c} \ell_y(t) |y\rangle. \quad (17)$$

Iterations of the form (15) will be applied to an arbitrary initial distribution of amplitudes $|g(0)\rangle$. Any single iteration changes amplitudes according to the equations

$$k_x(t+1) = \langle x|\mathbf{G}|g(t)\rangle, \quad \ell_y(t+1) = \langle y|\mathbf{G}|g(t)\rangle,$$

where $x \in \mathcal{M}$ and $y \in \mathcal{M}^c$. These equations are accompanied by initial amplitudes $k_x(0)$ and $\ell_y(0)$ of marked and unmarked items, respectively. The authors of [10] have examined the above recursion equations. Their results required for us are summarized in Appendix A. We will use them to exemplify relations between quantum coherence and the success probability in amplitude amplification.

III. GENERAL RELATIONS BETWEEN QUANTUM COHERENCE AND THE SUCCESS PROBABILITY

In this section, we focus on general relations that enlighten the role of coherence changes in amplitude amplification. In practice, computing devices are inevitably exposed to noise. In this situation, the state of a quantum register after t steps is described by density matrix $\rho(t)$. Then the probability to measure one of the marked state is written as

$$P_{\text{suc}}(t) = \sum_{x \in \mathcal{M}} \langle x|\rho(t)|x\rangle. \quad (18)$$

Amplitude amplification technique aims to magnify amplitudes of desired states unknown *a priori*. Each good algorithm of the considered type should provide such an amplification. In order to increase the success probability $P_{\text{suc}}(t)$, a coherence of the register should be used somehow. For the original formulation of Grover's algorithm, this issue was addressed in [17]. We will analyze the problem in more general setting. In addition, the geometric coherence was not utilized in this context.

Using the geometric coherence, we observe its very immediate link with the probability of success. In the case of pure states, the geometric coherence is expressed by (9). We first suppose that only one amplitude should be maximized a particular problem of amplitude amplification. This situation is sufficiently typical, but not general. For a time, we also assume that states of a quantum register are pure during the performance of algorithm. After a proper number of algorithm steps, amplitudes in some superposition $|\psi\rangle$ should become sufficiently small, except for the unique one. Hence, we have observed a direct complementarity relation,

$$C_g(|\psi\rangle) + P_{\text{suc}}(|\psi\rangle) = 1. \quad (19)$$

In other words, under accepted circumstances increase of the success probability implies decreasing the geometric coherence, and *vice versa*. Thus, quantum coherence with respect to the calculation basis seems to be a key resource in amplitude amplification. In the form of inequality, we can exceed the above relation to mixed states of a quantum register and arbitrary number of marked states.

Proposition 1 *Let ρ be a density matrix normalized as $\text{tr}(\rho) = 1$, and let $P_{\text{suc}}(\rho)$ be defined according to (18). The geometric coherence of ρ satisfies the inequality*

$$C_g(\rho) + \frac{1}{M} P_{\text{suc}}(\rho) \leq 1. \quad (20)$$

Proof. Let us consider the diagonal state

$$\delta_0 = \frac{1}{M} \sum_{x \in \mathcal{M}} |x\rangle\langle x|.$$

Using the property P4(b) of [30], we can write the inequalities

$$\max_{\delta \in \mathcal{I}} F(\delta, \rho) \geq F(\delta_0, \rho) \geq \text{tr}(\delta_0 \rho) = \frac{P_{\text{suc}}(\rho)}{M}. \quad (21)$$

Combining (8) with (21) completes the proof. ■

Returning to the case of amplitude amplification, we see the following. If the success probability is determined by the unique state of the calculation basis, i.e., $M = 1$, then

$$C_g(\rho) + P_{\text{suc}}(\rho) \leq 1. \quad (22)$$

In reality, states of the quantum register are inevitably exposed to noise. With some amount even small, they will become mixed. So, the trade-off relation between quantum coherence and the success probability may only be restricted here. It should be noted that the geometric coherence is not a coherence measure in the sense that is proposed by the authors of [15]. They added the list of axioms for coherence quantifiers by two items called the uniqueness for pure states and the additivity. In this regard, we are also interested in relations of the success probability with other coherence quantifiers.

The coherence quantifier based on the relative entropy is one of the most natural measure for these purposes. At the same time, this measure is not connected with P_{suc} so immediately as the geometric coherence. Here, the following statement holds.

Proposition 2 *Let ρ be a density matrix normalized as $\text{tr}(\rho) = 1$, and let $P_{\text{suc}}(\rho)$ be defined according to (18). The relative entropy of coherence satisfies the inequalities*

$$h_1(P_{\text{suc}}) \leq C_1(\rho) + S(\rho) \leq P_{\text{suc}} \ln\left(\frac{M}{P_{\text{suc}}}\right) + (1 - P_{\text{suc}}) \ln\left(\frac{N - M}{1 - P_{\text{suc}}}\right), \quad (23)$$

where $M = |\mathcal{M}|$ and $h_1(P_{\text{suc}})$ is the binary Shannon entropy. If the state ρ is such that

$$\arg(\max p(x)) \in \mathcal{M}, \quad (24)$$

where $p(x) = \langle x | \rho | x \rangle$, then we also have

$$-\ln P_{\text{suc}} \leq C_1(\rho) + S(\rho). \quad (25)$$

Proof. To prove the right-hand side of (23), we will use the results of [31]. The authors of [31] addressed the situation, when independent events are somehow collected into nonempty and pairwise disjoint subsets. Upper bounds on the Shannon entropy of the original probability distribution is then expressed in terms of new probabilities corresponding to these subsets. In our case, we have the subsets \mathcal{M} and \mathcal{M}^c with probabilities P_{suc} and $1 - P_{\text{suc}}$, respectively. Applying theorem 3 of [31] to the probabilities $p(x)$, we write

$$H_1(p) \leq P_{\text{suc}} \ln\left(\frac{|\mathcal{M}|}{P_{\text{suc}}}\right) + (1 - P_{\text{suc}}) \ln\left(\frac{|\mathcal{M}^c|}{1 - P_{\text{suc}}}\right).$$

Combining this with the definition of $C_1(\rho)$ completes the proof of the right-hand side of (23).

Let us proceed to the left-hand side of (23). We first observe that

$$\frac{1}{P_{\text{suc}}} \sum_{x \in \mathcal{M}} p(x) = 1, \quad \frac{1}{1 - P_{\text{suc}}} \sum_{y \in \mathcal{M}^c} p(y) = 1.$$

Applying Jensen's inequality to convex function $\xi \mapsto -\ln \xi$ then gives

$$H_1(p) \geq -P_{\text{suc}} \ln\left(\frac{1}{P_{\text{suc}}} \sum_{x \in \mathcal{M}} p(x)^2\right) - (1 - P_{\text{suc}}) \ln\left(\frac{1}{1 - P_{\text{suc}}} \sum_{y \in \mathcal{M}^c} p(y)^2\right). \quad (26)$$

For all $x \in \mathcal{M}$ and $y \in \mathcal{M}^c$, we have $p(x) \leq P_{\text{suc}}$ and $p(y) \leq 1 - P_{\text{suc}}$, whence

$$\sum_{x \in \mathcal{M}} p(x)^2 \leq P_{\text{suc}}^2, \quad \sum_{y \in \mathcal{M}^c} p(y)^2 \leq (1 - P_{\text{suc}})^2.$$

Combining the latter with (26) and decreasing of the function $\xi \mapsto -\ln \xi$ finally gives $H_1(p) \geq h_1(P_{\text{suc}})$.

The proof of (25) begins with applying Jensen's inequality to convex function $\xi \mapsto -\ln \xi$, so that

$$-\sum_{x \in \mathcal{M}} p(x) \ln p(x) \geq -\ln\left(\sum_{x=0}^{N-1} p(x)^2\right). \quad (27)$$

Under the condition (24), we write $p(x) \leq \max p(x) \leq P_{\text{suc}}$ for all $x \in \{0, 1, \dots, N-1\}$, whence

$$\sum_{x=0}^{N-1} p(x)^2 \leq P_{\text{suc}}.$$

Together with (27) and decreasing of $\xi \mapsto -\ln \xi$, the last inequality provides (25). ■

The statement of Proposition 2 gives a two-sided estimate of the relative entropy of coherence in terms of the success probability $P_{\text{suc}}(\rho)$. When this probability approaches 1, a band of allowed values of $C_1(\rho)$ becomes more and more narrow. Of course, such a picture relates to good algorithms, which ensure high chances for the success. In a general scenario of amplitude amplification, we have herewith seen complementarity between coherence and the success probability. For the fixed $P_{\text{suc}}(\rho)$, a difference between the lower and upper bounds mainly depends on the ratio M/N . The less this ratio is, the less width of coherence variations is given by the two-sided estimate (23).

The following observation concerns the presented upper bound. Inspecting the sign of derivative, we see that the right-hand side of (23) decreases with P_{suc} on the interval $(M/N; 1)$. The value M/N gives the success probability in the trivial algorithm of random choice of items, when no amplification actually takes place. Over this value, any increase of the success probability will lead to decreasing range of allowed changes of quantum coherence with respect to the calculation basis. To reach sufficiently high values of P_{suc} , an algorithm of amplitude amplification should somehow provide a coherence depletion. Note also that the right-hand side of (23) is saturated in some examples of amplitude amplification. Thus, this upper bound cannot be improved without including additional parameters into a consideration.

We have also obtained two lower bounds on the relative entropy of coherence. The inequality (25) holds under the additional condition (24). For $P_{\text{suc}} < 1/2$, it provides a stronger bound on the relative entropy of coherence. In principle, the condition (24) is sufficiently natural for a good technique of amplitude amplification. It may also be caused by prior evidences recorded in the initial amplitude distribution. On the other hand, we usually wish to have algorithms that provide values $P_{\text{suc}} > 1/2$ and rather as closely to 1 as possible. In this case, the left-hand side of (23) gives a better estimate from below

Among distance-based coherence quantifiers, the ℓ_1 -norm of coherence is one of most intuitive [14]. On the other hand, it does not fulfill additional axioms proposed in [15]. We have observed that general trade-off relations between $C_{\ell_1}(\rho)$ and $P_{\text{suc}}(\rho)$ are sufficiently difficult to formulate. Nevertheless, certain conclusions can be obtained for some assumed forms of the state of quantum register. In this regard, our consideration is an extension of the corresponding discussion given in [17]. Let us consider the superposition

$$|\psi\rangle = \sum_{x=0}^{N-1} c_x |x\rangle, \quad (28)$$

We first suppose that amplitudes has the same absolute values separately for labels in \mathcal{M} and \mathcal{M}^c , namely

$$|c_x| = \alpha \quad \forall x \in \mathcal{M}, \quad |c_y| = \beta \quad \forall y \in \mathcal{M}^c.$$

Such amplitude distribution is typical during a performance of the standard Grover algorithm. In this case, we easily obtain $\alpha^2 = P_{\text{suc}}/M$ and $\beta^2 = (1 - P_{\text{suc}})/(N - M)$. Combining the latter with the expression

$$1 + C_{\ell_1}(|\psi\rangle) = \sum_{x,y=0}^{N-1} |c_x c_y|,$$

we finally obtain

$$C_{\ell_1}(|\psi\rangle) = \left(\sqrt{MP_{\text{suc}}} + \sqrt{(N-M)(1-P_{\text{suc}})} \right)^2 - 1. \quad (29)$$

At first glance, a complementarity between the coherence quantifier and the success probability is not obvious from (29). Following [17], we now take very natural assumption $M \ll N$. Then we can convert (29) into a direct complementarity relation,

$$C_{\ell_1}(|\psi\rangle) + NP_{\text{suc}} = N[1 + O(M/N)]. \quad (30)$$

Rescaling by the denominator N , this relation becomes very similar to (19) and (22). This rescaling could be expected here, since the ℓ_1 -norm of coherence can reach values up to $N - 1$ [25], whereas the geometric coherence cannot exceed 1. In the context of original Grover's algorithm, the relation (30) was presented in [17]. We only note that the result (30) reflects no more than a "boxcar" distribution of amplitudes.

It is instructive to address the case, in which only one of desired states has relatively large amplitude. So, we suppose that $|c_{x_0}| = \alpha$ for the unique $x_0 \in \mathcal{M}$ and $|c_x| = \beta$ for all other x in both the sets \mathcal{M} and \mathcal{M}^c . Situations of such a kind may occur in amplitude amplification with *a priori* knowledge, for instance, in the initial amplitude distribution. Here, we have

$$P_{\text{suc}} = \alpha^2 + (M - 1)\beta^2, \quad 1 - P_{\text{suc}} = (N - M)\beta^2. \quad (31)$$

Calculations of the ℓ_1 -norm of coherence finally gives

$$C_{\ell_1}(|\psi\rangle) = \frac{1}{N-M} \left(\sqrt{(N-1)P_{\text{suc}} - (M-1)} + (N-1)\sqrt{1-P_{\text{suc}}} \right)^2 - 1. \quad (32)$$

Assuming $M \ll N$ again, we have arrived at the approximate formula

$$C_{\ell_1}(|\psi\rangle) = \sqrt{4P_{\text{suc}}(1-P_{\text{suc}})} + O(M/N). \quad (33)$$

This relation differs from (30) in a style. Nevertheless, the following fact can be observed. We usually expect to reach values $P_{\text{suc}} > 1/2$. In the interval $P_{\text{suc}} \in (1/2; 1)$, the right-hand side of (33) decreases with growth of P_{suc} . Thus, the ℓ_1 -norm of coherence and the success probability should be mutually complementary, at least for good algorithms with high chances for success. On the other hand, the last case gives an evidence that general complementarity relations for the ℓ_1 -norm of coherence are difficult to formulate. Despite of very simple structure of the density matrix, the expression (32) is complicated enough. It seems that other coherence quantifiers should be preferred, when we concern the role of quantum coherence in amplitude amplification.

IV. SOME MODEL SCENARIOS OF COHERENCE CHANGES IN AMPLITUDE AMPLIFICATIONS

In this section, we will illustrate trade-off relations between coherence and the success probability within explicit model examples of amplitude amplification. We mainly focus on the relative entropy of coherence, since it seems to be more sensitive to parameter variations than the geometric coherence. We also aim to emphasize distinctions between cases, when marked and unmarked states are dealt with consistently or inconsistently. For convenience, we begin with inspecting the question in original Grover's formulation [1].

A. Original Grover's formulation

The initial amplitude distribution is taken to be (10), and the rotation angles are $\beta = \gamma = \pi$. By $M = |\mathcal{M}|$, we mean the number of marked states, then $N - M = |\mathcal{M}^c|$. For all $x \in \mathcal{M}$ and $y \in \mathcal{M}^c$, the initial amplitudes appear as

$$k_x(0) = \ell_y(0) = \frac{1}{\sqrt{N}}. \quad (34)$$

For the original algorithm, we have $|\eta\rangle = \mathbf{H}|0\rangle$ and $\eta_x = \eta_y = 1/\sqrt{N}$, whence

$$W_k = \frac{M}{N}, \quad W_\ell = \frac{N-M}{N}. \quad (35)$$

Further, we at once obtain

$$k'_x(t) = \sqrt{N} k_x(t), \quad \ell'_y(t) = \sqrt{N} \ell_y(t), \quad (36)$$

including $k'_x(0) = \ell'_x(0) = 1$. In line with (A3) and (A4), the initial weighted averages appear as $\tilde{k}'(0) = \tilde{\ell}'(0) = 1$, so that

$$\Delta k'_x = \Delta \ell'_y = 0. \quad (37)$$

Then the formulas (A12) and (A13) merely say that $k'_x(t) = \tilde{k}'_x(t)$ and $\ell'_x(t) = \tilde{\ell}'_x(t)$ for all t . In the original formulation, we actually deal only with two different values of amplitudes.

Substituting $\beta = \gamma = \pi$ together with (35) into (A6), we get

$$\cos \omega = W_\ell - W_k = \frac{N-2M}{N}. \quad (38)$$

Recall that we assume $1 \leq M \leq N/2$. So, the parameter ω is defined by (A6) and $\omega \in (0; \pi/2]$. It will be convenient to remember the formulas

$$\sin^2 \omega/2 = \frac{M}{N}, \quad \cos^2 \omega/2 = 1 - \frac{M}{N}. \quad (39)$$

For $\beta = \gamma = \pi$, we further have $\omega_{\pm} = 2\pi \pm \omega$ due to (A5), whence the eigenvalues (A5) read

$$\lambda_{\pm} = e^{\pm i\omega} = \cos \omega \pm i \sin \omega.$$

Further calculations lead to the following expressions,

$$\xi_1 = \frac{-i \exp(+i\omega/2)}{2 \sin \omega/2}, \quad \xi_2 = \frac{-i \exp(-i\omega/2)}{2 \sin \omega/2}. \quad (40)$$

Using (A9) together with $b = 1 + \cos \omega$ and (40), we obtain

$$\xi_3 = \frac{\exp(+i\omega/2)}{2 \cos \omega/2}, \quad \xi_4 = \frac{-\exp(-i\omega/2)}{2 \cos \omega/2}. \quad (41)$$

Since t is integer, we may take $\omega_{\pm} = \pm \omega$ instead of $\omega_{\pm} = 2\pi \pm \omega$. Due to (A7), one has

$$\tilde{k}'(t) = \frac{\sin[\omega(t+1/2)]}{\sin \omega/2}, \quad \tilde{\ell}'(t) = \frac{\cos[\omega(t+1/2)]}{\cos \omega/2}. \quad (42)$$

As was already mentioned, $k'_x(t) = \tilde{k}'(t)$ and $\ell'_y(t) = \tilde{\ell}'(t)$. Thus, we finally write

$$k_x(t) = \frac{1}{\sqrt{N}} \frac{\sin[\omega(t+1/2)]}{\sin \omega/2}, \quad \ell_y(t) = \frac{1}{\sqrt{N}} \frac{\cos[\omega(t+1/2)]}{\cos \omega/2},$$

for all $x \in \mathcal{M}$ and $y \in \mathcal{M}^c$. The probabilities of interest appear as

$$P_{\text{suc}}(t) = \sum_{x \in \mathcal{M}} |k_x(t)|^2 = \frac{M}{N \sin^2 \omega/2} \sin^2[\omega(t+1/2)] = \sin^2[\omega(t+1/2)], \quad (43)$$

$$1 - P_{\text{suc}}(t) = \sum_{y \in \mathcal{M}^c} |\ell_y(t)|^2 = \frac{N-M}{N \cos^2 \omega/2} \cos^2[\omega(t+1/2)] = \cos^2[\omega(t+1/2)]. \quad (44)$$

We wish to relate these probabilities with the coherence quantifiers. Since the state of the register is pure, its von Neumann entropy is zero. Further, the diagonal part of the density matrix appears as

$$\rho_{\text{diag}} = \text{diag} \left(\overbrace{|k_x(t)|^2, \dots, |k_x(t)|^2}^{M \text{ entries}}, \underbrace{|\ell_y(t)|^2, \dots, |\ell_y(t)|^2}_{N-M \text{ entries}} \right). \quad (45)$$

Hence, the geometric coherence obeys the complementarity relation

$$C_g(|g(t)\rangle) = 1 - \frac{P_{\text{suc}}(t)}{M}, \quad (46)$$

whenever $|k_x(t)|^2 \geq |\ell_y(t)|^2$. Otherwise, the geometric coherence is equal to the 1 minus $(1 - P_{\text{suc}}(t))/(N - M)$. Further, the relative entropy of coherence is calculated as

$$C_1(|g(t)\rangle) = -M|k_x(t)|^2 \ln(|k_x(t)|^2) - (N-M)|\ell_y(t)|^2 \ln(|\ell_y(t)|^2). \quad (47)$$

Due to (39), we can write

$$M|k_x(t)|^2 = \sin^2[\omega(t+1/2)] = P_{\text{suc}}(t), \quad (N-M)|\ell_y(t)|^2 = \cos^2[\omega(t+1/2)] = 1 - P_{\text{suc}}(t). \quad (48)$$

Combining (47) with (48) results in the important formula

$$C_1(|g(t)\rangle) = -P_{\text{suc}}(t) \ln\left(\frac{P_{\text{suc}}(t)}{M}\right) - (1 - P_{\text{suc}}(t)) \ln\left(\frac{1 - P_{\text{suc}}(t)}{N - M}\right). \quad (49)$$

This expression coincides with the right-hand side of (23). In opposite to (46), the formula (49) holds for all t . Therefore, the upper bound given by (23) is saturated in the original formulation of Grover's search. For the given P_{suc} , the relative entropy of coherence reaches the maximal value approved by the right-hand side of (23). Grover's search algorithm works so that any coherence decreasing is used in the most efficient way.

B. Marked and unmarked states are consistently amplified or decayed

In this subsection, we consider relations between coherence and the success probability, when the terms U and $|s\rangle$ are such that marked states are all amplified or decayed consistently. This is a particular situation, in which some prior knowledge is provided. More precisely, the state $|\eta\rangle$ is assumed to have the amplitudes

$$\eta_x = \sqrt{\frac{M_\eta}{MN}}, \quad \eta_y = \sqrt{\frac{N - M_\eta}{(N - M)N}}, \quad (50)$$

where $x \in \mathcal{M}$ and $y \in \mathcal{M}^c$. The angles β and γ remain the same as in the original formulation. When amplitudes are balanced with respect to both \mathcal{M} and \mathcal{M}^c , we can describe them by a single parameter, $M_\eta > 0$. Generally, this parameter is not integer. An efficiency is increased with $M_\eta > M$, though we will also address $M_\eta < M$. Here, we replace (35) with the formulas

$$W_k = \frac{M_\eta}{N}, \quad W_\ell = \frac{N - M_\eta}{N}. \quad (51)$$

Combining (50) with (A3) and (A4) directly gives $k'_x(t) = \tilde{k}'(t)$ for $x \in \mathcal{M}$ and $\ell'_y(t) = \tilde{\ell}'(t)$ for $y \in \mathcal{M}^c$. Rewriting (A6) with M_η instead of M , we have the parameter ω_η such that

$$\sin^2 \omega_\eta / 2 = \frac{M_\eta}{N}, \quad \cos^2 \omega_\eta / 2 = 1 - \frac{M_\eta}{N}. \quad (52)$$

The latter should be used simultaneously with (39). The initial amplitude distribution (34) is the same as in the original formulation. Correspondingly to (50), we have

$$k'_x(t) = \sqrt{\frac{MN}{M_\eta}} k_x(t), \quad \ell'_y(t) = \sqrt{\frac{(N - M)N}{N - M_\eta}} \ell_y(t). \quad (53)$$

Together with $k_x(0) = \ell_y(0) = 1/\sqrt{N}$, one then obtains

$$k'_x(0) = \tilde{k}'(0) = \sqrt{\frac{M}{M_\eta}} = \frac{\sin \omega / 2}{\sin \omega_\eta / 2}, \quad (54)$$

$$\ell'_x(0) = \tilde{\ell}'(0) = \sqrt{\frac{N - M}{N - M_\eta}} = \frac{\cos \omega / 2}{\cos \omega_\eta / 2}, \quad (55)$$

where the parameter ω is again defined by (38).

For $\beta = \gamma = \pi$, we further have $\omega_\pm = 2\pi \pm \omega_\eta$. As calculations show, the formulas (40) are (41) are replaced with

$$\xi_1 = \frac{-i \exp(+i\omega/2)}{2 \sin \omega_\eta / 2}, \quad \xi_2 = \frac{-i \exp(-i\omega/2)}{2 \sin \omega_\eta / 2}, \quad (56)$$

$$\xi_3 = \frac{\exp(+i\omega/2)}{2 \cos \omega_\eta / 2}, \quad \xi_4 = \frac{-\exp(-i\omega/2)}{2 \cos \omega_\eta / 2}. \quad (57)$$

Similarly to (42), we obtain the averaged amplitudes

$$\tilde{k}'(t) = \frac{\sin(\omega_\eta t + \omega/2)}{\sin \omega_\eta / 2}, \quad \tilde{\ell}'(t) = \frac{\cos(\omega_\eta t + \omega/2)}{\cos \omega_\eta / 2}. \quad (58)$$

These formulas also represent the amplitudes $k'_x(t) = \tilde{k}'(t)$ for $x \in \mathcal{M}$ and $\ell'_y(t) = \tilde{\ell}'(t)$ for $y \in \mathcal{M}^c$. Substituting $t = 0$, they reduce to (54) and (55). Combining (53) with (58) immediately gives

$$|k_x(t)|^2 = \frac{\sin^2(\omega_\eta t + \omega/2)}{M}, \quad |\ell_y(t)|^2 = \frac{\cos^2(\omega_\eta t + \omega/2)}{N - M}.$$

Calculating the probabilities then results in

$$P_{\text{suc}}(t) = \sin^2(\omega_\eta t + \omega/2), \quad 1 - P_{\text{suc}}(t) = \cos^2(\omega_\eta t + \omega/2). \quad (59)$$

When $|k_x(t)|^2 \geq |\ell_y(t)|^2$, the geometric coherence obeys (46). We also note that relations (48) are still valid. The relative entropy of coherence is again connected with the probabilities by (49). The success probability is first maximized, when $\omega_\eta t + \omega/2$ becomes as close to $\pi/2$ as possible. Since t should be integer, we will take one of the two numbers

$$\left\lfloor \frac{\pi - \omega}{2\omega_\eta} \right\rfloor, \quad \left\lceil \frac{\pi - \omega}{2\omega_\eta} \right\rceil. \quad (60)$$

With growth of $M_\eta > M$ in (50), the parameter $\omega_\eta > \omega$ also increases. For the fixed M and N , the maximum of the success probability will be reached in lesser number of iterations due to conducive prior knowledge. When $M_\eta < M$, the situation is opposite. As prior knowledge now prevents, reaching the maximum of the success probability will demand more iterations. In both the situations, nevertheless, the quantity $C_1(|g(t)\rangle)$ decreases with increasing $P_{\text{suc}}(t)$ along the right-hand side of (23). Similarly to the original formulation, any coherence reducing is used mostly efficiently. This feature holds due to the consistency during the computing process. It evolves so that states are evenly amplified in \mathcal{M} and attenuated in \mathcal{M}^c .

C. Consistent prior knowledge contained in the initial distribution

In this subsection, we address another case, in which the initial amplitude distribution is not uniform. Concerning this situation, some results were reported in [17]. We will focus on those aspects that are not addressed therein. Let us consider the consistent initial distribution such that

$$k_x(0) = \sqrt{\frac{M_0}{MN}}, \quad \ell_y(0) = \sqrt{\frac{N - M_0}{(N - M)N}}. \quad (61)$$

Here, the parameter $M_0 > 0$ is not integer and may deviate from M in both sides. For $\eta_x = \eta_y = 1/\sqrt{N}$, the weights W_k and W_ℓ are defined by (51). Also, the rescaled initial amplitudes are written as (36). Similarly to (52), we put the parameter ω_0 such that

$$\sin^2 \omega_0/2 = \frac{M_0}{N}, \quad \cos^2 \omega_0/2 = 1 - \frac{M_0}{N}. \quad (62)$$

Hence, we write the rescaled initial amplitudes

$$k'_x(0) = \tilde{k}'(0) = \sqrt{\frac{M_0}{M}} = \frac{\sin \omega_0/2}{\sin \omega/2}, \quad (63)$$

$$\ell'_x(0) = \tilde{\ell}'(0) = \sqrt{\frac{N - M_0}{N - M}} = \frac{\cos \omega_0/2}{\cos \omega/2}, \quad (64)$$

We also have $\omega_\pm = 2\pi \pm \omega$ just as in the original formulation. So, we merely substitute the new initial amplitudes (63) and (64) into the previous formulas. The resulting expressions read

$$\xi_1 = \frac{-i \exp(+i\omega_0/2)}{2 \sin \omega/2}, \quad \xi_2 = \frac{-i \exp(-i\omega_0/2)}{2 \sin \omega/2}, \quad (65)$$

$$\xi_3 = \frac{\exp(+i\omega_0/2)}{2 \cos \omega/2}, \quad \xi_4 = \frac{-\exp(-i\omega_0/2)}{2 \cos \omega/2}. \quad (66)$$

Similarly to (42) and (58), the averaged amplitudes are written as

$$\tilde{k}'(t) = \frac{\sin(\omega t + \omega_0/2)}{\sin \omega/2}, \quad \tilde{\ell}'(t) = \frac{\cos(\omega t + \omega_0/2)}{\cos \omega/2}.$$

These expressions also give the amplitudes $k'_x(t) = \tilde{k}'(t)$ for $x \in \mathcal{M}$ and $\ell'_x(t) = \tilde{\ell}'(t)$ for $y \in \mathcal{M}^c$. Taking into account (39), one then obtains

$$|k_x(t)|^2 = \frac{\sin^2(\omega t + \omega_0/2)}{M}, \quad |\ell_y(t)|^2 = \frac{\cos^2(\omega t + \omega_0/2)}{N - M}. \quad (67)$$

Similarly to (59), we now write the probabilities

$$P_{\text{suc}}(t) = \sin^2(\omega t + \omega_0/2), \quad 1 - P_{\text{suc}}(t) = \cos^2(\omega t + \omega_0/2). \quad (68)$$

When $|k_x(t)|^2 \geq |\ell_y(t)|^2$, for the geometric coherence we again have (46). The success probability is first maximized for t equal to one of the two numbers

$$\left\lfloor \frac{\pi - \omega_0}{2\omega} \right\rfloor, \quad \left\lceil \frac{\pi - \omega_0}{2\omega} \right\rceil. \quad (69)$$

With growth of $M_0 > M$ in (50), the term $\omega_0 > \omega$ also increases. It may reduce a number of iterations due to prior knowledge in prepared initial distribution of amplitudes. The situation is opposite for $M_0 < M$, when reaching the maximum of the success probability demand more iterations. In comparison with (60), the numbers (69) are changed by means of ω_0 in the numerator. In both the situations, $\omega_0 > \omega$ and $\omega_0 < \omega$, the relation between $C_1(|g(t)\rangle)$ and $P_{\text{suc}}(t)$ again follows the line given by (49). In other words, an amplification process is balanced so that the right-hand side of (23) is saturated. This claim easily follows from (67). We have again seen the role of consistency in amplitude amplification. Indeed, states are evenly amplified in \mathcal{M} and attenuated in \mathcal{M}^c .

D. Marked and unmarked states are inconsistently amplified or decayed

Finally, we consider an unbalanced case, in which consistency of states in amplitude amplification is broken. Suppose that the amplitudes of the state $|\eta\rangle$ can take only two different values,

$$\sqrt{\frac{1+\alpha}{N}}, \quad \sqrt{\frac{1-\alpha}{N}}, \quad (70)$$

where $\alpha \in [0; 1]$. We further assume that M is even and the values (70) are distributed as follows. For items $x \in \mathcal{M}$, there are $M/2$ amplitudes $\eta_x = \sqrt{(1+\alpha)/N}$ and $M/2$ amplitudes $\eta_x = \sqrt{(1-\alpha)/N}$. In effect, we also put $\eta_y = \sqrt{(1+\alpha)/N}$ for one half and $\eta_y = \sqrt{(1-\alpha)/N}$ for other half of items $y \in \mathcal{M}^c$. Then the weights W_k and W_ℓ satisfy (35).

For the uniform initial distribution, the rescaled initial amplitudes are written as

$$k'_x(0) = \frac{1}{\sqrt{1 \pm \alpha}}, \quad \ell'_y(0) = \frac{1}{\sqrt{1 \pm \alpha}}.$$

By substituting, the initial averaged amplitudes then read

$$\tilde{k}'(0) = \tilde{\ell}'(0) = \frac{\sqrt{1+\alpha} + \sqrt{1-\alpha}}{2}.$$

Unlike the above balanced cases, the initial differences take non-zero values, namely

$$\Delta k'_x = \Delta \ell'_y = \frac{1}{\sqrt{1 \pm \alpha}} - \frac{\sqrt{1+\alpha} + \sqrt{1-\alpha}}{2}. \quad (71)$$

Since $\tilde{k}'(0) = \tilde{\ell}'(0)$, we can obtain the coefficients $\xi_1, \xi_2, \xi_3, \xi_4$ as follows. The expressions (40) and (41) should all be multiplied by the factor $(\sqrt{1+\alpha} + \sqrt{1-\alpha})/\sqrt{2}$. Hence, the averaged amplitudes become

$$\begin{aligned} \tilde{k}'(t) &= \frac{\sqrt{1+\alpha} + \sqrt{1-\alpha}}{2} \frac{\sin[\omega(t+1/2)]}{\sin \omega/2}, \\ \tilde{\ell}'(t) &= \frac{\sqrt{1+\alpha} + \sqrt{1-\alpha}}{2} \frac{\cos[\omega(t+1/2)]}{\cos \omega/2}. \end{aligned}$$

Hence, we obtain the amplitudes of marked and unmarked states in the form

$$k_x(t) = \sqrt{\frac{1 \pm \alpha}{N}} \left(\tilde{k}'(t) + \frac{1}{\sqrt{1 \pm \alpha}} - \frac{\sqrt{1+\alpha} + \sqrt{1-\alpha}}{2} \right), \quad (72)$$

$$\ell_y(t) = \sqrt{\frac{1 \pm \alpha}{N}} \left(\tilde{\ell}'(t) + \frac{(-1)^t}{\sqrt{1 \pm \alpha}} - \frac{\sqrt{1+\alpha} + \sqrt{1-\alpha}}{2(-1)^t} \right). \quad (73)$$

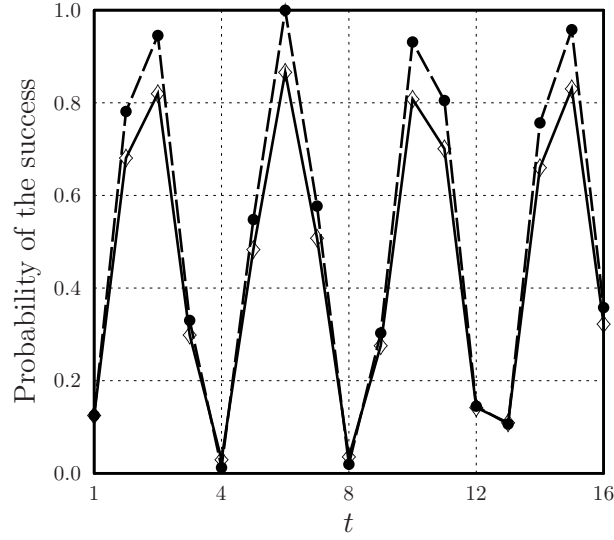


FIG. 1: $P_{\text{suc}}(t)$ as a function of integer t is shown by solid line for $\alpha = 0.72$ and by dashed line for $\alpha = 0$. Both the lines are related to the case $N = 16$ and $M = 2$.

For each of the lines (72) and (73), we have the sign plus and the sign minus exactly for one half of the amplitudes. Doing some calculations, the desired probabilities are finally expressed as follows. For $\varepsilon = \pm$, we introduce the quantities

$$P^{(\varepsilon)}(t) = \frac{1}{2} \left[\sin \frac{\omega}{2} + (1 + \varepsilon\alpha + \sqrt{1 - \alpha^2}) \sin \frac{\omega t}{2} \cos \frac{\omega(t+1)}{2} \right]^2, \quad (74)$$

$$Q^{(\varepsilon)}(t) = \frac{1}{2} \left[\cos \frac{\omega}{2} - (1 + \varepsilon\alpha + \sqrt{1 - \alpha^2}) \sin \frac{\omega t}{2} \sin \frac{\omega(t+1)}{2} \right]^2, \quad (75)$$

$$Q^{(\varepsilon)}(t) = \frac{1}{2} \left[\cos \frac{\omega}{2} - (1 + \varepsilon\alpha + \sqrt{1 - \alpha^2}) \cos \frac{\omega t}{2} \cos \frac{\omega(t+1)}{2} \right]^2, \quad (76)$$

where (75) stands for even t and (76) stands for odd t . In terms of these quantities, the probabilities of interest are finally expressed as

$$P_{\text{suc}}(t) = P^{(+)}(t) + P^{(-)}(t), \quad 1 - P_{\text{suc}}(t) = Q^{(+)}(t) + Q^{(-)}(t). \quad (77)$$

Further, the relative entropy of coherence is represented as

$$C_1(|g(t)\rangle) = \sum_{\varepsilon=\pm} P^{(\varepsilon)}(t) \ln \left(\frac{M/2}{P^{(\varepsilon)}(t)} \right) + \sum_{\varepsilon=\pm} Q^{(\varepsilon)}(t) \ln \left(\frac{(N-M)/2}{Q^{(\varepsilon)}(t)} \right). \quad (78)$$

Since the action of (16) is not balanced, the relation between $C_1(|g(t)\rangle)$ and $P_{\text{suc}}(t)$ does not follow (49). Due to an inconsistency in amplitude amplification, coherence reducing is not used in the most efficient way. A similar picture occurs with inconsistent prior knowledge in the initial distribution. We refrain from presenting the details here.

To illustrate the above conclusions, we now visualize $C_1(|g(t)\rangle)$ versus t for a some simple choice of parameters. Since the relative entropy of coherence satisfies

$$\max\{h_1(P_{\text{suc}}(t)), -\ln P_{\text{suc}}(t)\} \leq C_1(|g(t)\rangle) \leq P_{\text{suc}}(t) \ln \left(\frac{M}{P_{\text{suc}}(t)} \right) + (1 - P_{\text{suc}}(t)) \ln \left(\frac{N-M}{1 - P_{\text{suc}}(t)} \right), \quad (79)$$

we will also present both the sides of this two-sided estimation. It must be stressed that the lower bound (25) is valid under the condition (24). With changes of t , the condition (24) will be violated from time to time. Then we disregard (25) and use the lower bound $h_1(P_{\text{suc}}(t))$ solely.

In Figs. 1 and 2, we visualize some quantities for the case $N = 16$, $M = 2$, and $\alpha = 0.72$. In Fig. 1, the success probability is shown as a function of t . For comparison, the success probability is also given in original formulation, when $\alpha = 0$. We see that an inconsistency of amplitudes of the state $|\eta\rangle$ leads to some decreasing of the probability

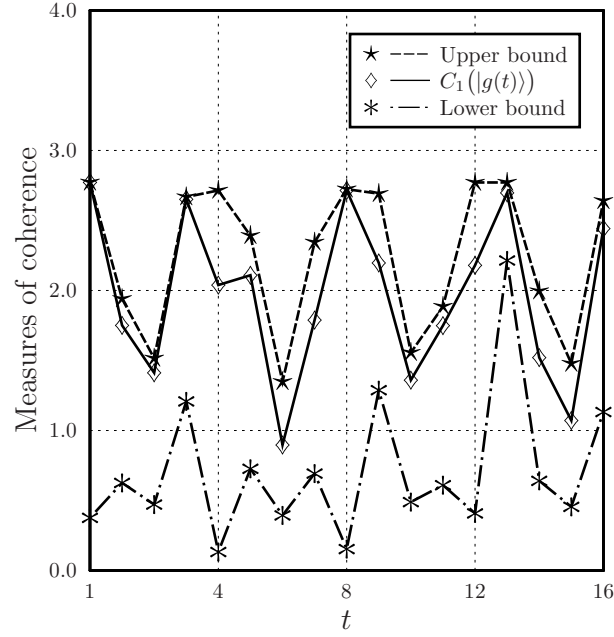


FIG. 2: The relative entropy of coherence $C_1(|g(t)\rangle\rangle$ and bounds on it as functions of t for the case $N = 16$, $M = 2$, and $\alpha = 0.72$.

peaks. On the other hand, reaching one or another peak requires the same number of iteration as in the original formulation. We also observe that the trade-off line between $C_1(|g(t)\rangle\rangle$ and the success probability does not follow the upper bound exactly. Dealing with marked and unmarked states inconsistently, this form of amplitude amplification cannot always use coherence changes in the most efficient way. Using Fig. 2, we can also estimate a quality of bounds in the two-sided estimate (23). As explicit general bounds, they seem to be sufficiently good.

V. CONCLUSIONS

We have examined the role of dealing with quantum coherence in amplitude amplification processes. General trade-off relations between quantum coherence and the success probability were derived. It seems that the geometric coherence and the relative entropy of coherence are more convenient quantifiers in this context. In comparison with the geometric coherence, the relative entropy of coherence is recognized as more sensitive. Using the success probability as a key parameter, we obtained a two-sided estimate on the relative entropy of coherence. This result describes a band of allowed values of quantum coherence. Basic conclusions are supported by explicit consideration of several model scenarios of amplitude amplification. Coherence changes can be used in the most efficient way only when marked and unmarked states are dealt with consistently. In other words, the computing process does not distinguish between the marked states as well as between the unmarked ones. Then any coherence decreasing will amplify the success probability as much as possible. Otherwise, there is a difference between concrete marked states and, maybe, between concrete unmarked states. In this unbalanced case, the relative entropy of coherence does not reach always its maximal value approved by the two-sided estimate for the given value of the success probability.

Appendix A: Results of analyzing the recursion equations

In this section, we briefly recall some formulas for the amplitudes in a generalized version of Grover's search with arbitrary initial distribution [10]. The solution is expressed in terms of rescaled amplitudes

$$k'_x(t) = \eta_x^{-1} k_x(t), \quad \ell'_y(t) = \eta_y^{-1} \ell_y(t). \quad (\text{A1})$$

where the coefficients $\eta_z = \langle z | \eta \rangle$ are assumed to be nonzero for all $z \in \mathcal{M} \cup \mathcal{M}^c$. Introducing the weights

$$W_k = \sum_{x \in \mathcal{M}} |\eta_x|^2, \quad W_\ell = \sum_{y \in \mathcal{M}^c} |\eta_y|^2, \quad (\text{A2})$$

so that $W_k + W_\ell = \langle \eta | \eta \rangle = 1$, the authors of [10] defined weighted averages of rescaled amplitudes

$$\tilde{k}'(t) = \frac{1}{W_k} \sum_{x \in \mathcal{M}} |\eta_x|^2 k'_x(t), \quad (\text{A3})$$

$$\tilde{\ell}'(t) = \frac{1}{W_\ell} \sum_{y \in \mathcal{M}^c} |\eta_y|^2 \ell'_y(t). \quad (\text{A4})$$

Then the recursion equations can be converted into a single matrix equation. This matrix equation has been solved by diagonalizing some 2×2 matrix [10]. The eigenvalues of this matrix are expressed as

$$\lambda_\pm = e^{i\omega_\pm}, \quad \omega_\pm = \pi + \frac{\beta + \gamma}{2} \pm \omega, \quad (\text{A5})$$

where the parameter ω which obeys $0 \leq \omega \leq \pi$ and

$$\cos \omega = W_k \cos \frac{\beta + \gamma}{2} + W_\ell \cos \frac{\beta - \gamma}{2}. \quad (\text{A6})$$

Under assumptions $W_k \neq 0$ and $W_\ell \neq 0$ together with $\gamma \in (0, 2\pi)$, the matrix of interest is certainly diagonalizable. The averaged amplitudes are finally expressed as

$$\tilde{k}'(t) = \xi_1 e^{i\omega_+ t} - \xi_2 e^{i\omega_- t}, \quad (\text{A7})$$

$$\tilde{\ell}'(t) = \xi_3 e^{i\omega_+ t} - \xi_4 e^{i\omega_- t}, \quad (\text{A8})$$

where the coefficients are found as $a = (1 - e^{i\beta}) e^{i\gamma} W_k - e^{i\gamma}$, $b = (1 - e^{i\beta}) W_\ell$ and

$$\begin{aligned} \xi_1 &= \frac{(\lambda_- - a)\tilde{k}'(0) - b\tilde{\ell}'(0)}{\lambda_- - \lambda_+}, & \xi_2 &= \frac{(\lambda_+ - a)\tilde{k}'(0) - b\tilde{\ell}'(0)}{\lambda_- - \lambda_+}, \\ \xi_3 &= \frac{\lambda_+ - a}{b} \xi_1, & \xi_4 &= \frac{\lambda_- - a}{b} \xi_2. \end{aligned} \quad (\text{A9})$$

Introducing the initial differences

$$\Delta k'_x = k'_x(0) - \tilde{k}'(0), \quad (\text{A10})$$

$$\Delta \ell'_y = \ell'_y(0) - \tilde{\ell}'(0), \quad (\text{A11})$$

the solution is completed by the formulas [10]

$$k'_x(t) = \tilde{k}'(t) + (-1)^t e^{i\gamma t} \Delta k'_x, \quad (\text{A12})$$

$$\ell'_y(t) = \tilde{\ell}'(t) + (-1)^t \Delta \ell'_y. \quad (\text{A13})$$

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